

AUTOMORPHISM GROUPS OF SOME PURE BRAID GROUPS

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ABSTRACT. We find finite presentations for the automorphism group of the Artin pure braid group and the automorphism group of the pure braid group associated to the full monomial group.

1. INTRODUCTION

Let B_n be the Artin braid group, with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-2$, and $\sigma_j \sigma_i = \sigma_i \sigma_j$ for $|j-i| \geq 2$. It is well known from work of Dyer and Grossman [DG81] that the automorphism group of the braid group may be realized as $\text{Aut}(B_n) \cong \overline{B}_n \rtimes \mathbb{Z}_2$, where \overline{B}_n denotes B_n modulo its center and \mathbb{Z}_2 acts by taking generators to their inverses. In this paper, we find an explicit presentation for the automorphism group of the Artin pure braid, the kernel $P_n = \ker(B_n \rightarrow \Sigma_n)$ of the natural map from the braid group to the symmetric group. The pure braid group has generators

$$(1) \quad A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_i,$$

and relations

$$(2) \quad A_{r,s}^{-1} A_{i,j} A_{r,s} = \begin{cases} A_{i,j} & \text{if } i < r < s < j, \\ A_{i,j} & \text{if } r < s < i < j, \\ A_{r,j} A_{i,j} A_{r,j}^{-1} & \text{if } r < s = i < j, \\ A_{r,j} A_{s,j} A_{i,j} A_{s,j}^{-1} A_{r,j}^{-1} & \text{if } r = i < s < j, \\ [A_{r,j}, A_{s,j}] A_{i,j} [A_{r,j}, A_{s,j}]^{-1} & \text{if } r < i < s < j, \end{cases}$$

where $[u, v] = uvu^{-1}v^{-1}$ is the commutator. Birman [Bir75] is a general reference.

Let

$$F(\mathbb{C}, n) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ if } i \neq j\}$$

be the configuration space of n distinct ordered points in \mathbb{C} . The symmetric group Σ_n acts freely on $F(\mathbb{C}, n)$ by permuting coordinates. Let $C(\mathbb{C}, n) = F(\mathbb{C}, n)/\Sigma_n$ denote the orbit space, the configuration space of n distinct unordered points in \mathbb{C} . It is well known that $P_n = \pi_1(F(\mathbb{C}, n))$, $B_n = \pi_1(C(\mathbb{C}, n))$, and that these spaces are Eilenberg-Mac Lane spaces for these braid groups.

For $i \neq j$, let $H_{i,j} = \ker(x_i - x_j)$, and let $\mathcal{A}_n = \{H_{i,j} \mid 1 \leq i < j \leq n\}$ denote the braid arrangement in \mathbb{C}^n , consisting of the reflecting hyperplanes of the symmetric group Σ_n . The configuration space $F(\mathbb{C}, n) = M(\mathcal{A}_n) = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} H_{i,j}$ may be realized as the complement of the braid arrangement \mathcal{A}_n . The other pure braid groups we consider may be viewed as arising from an analogous construction.

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Let r be a natural number greater than or equal to 2. The complex hyperplane arrangement $\mathcal{A}_{r,n}$ in \mathbb{C}^n defined by the polynomial

$$(3) \quad Q_{r,n} = Q(\mathcal{A}_{r,n}) = x_1 \cdots x_n \prod_{1 \leq i < j \leq n} (x_i^r - x_j^r)$$

is known as a full monomial arrangement. The complement $M(\mathcal{A}_{r,n}) = \mathbb{C}^n \setminus Q_{r,n}^{-1}(0)$ may be realized as the orbit configuration space

$$F_\Gamma(\mathbb{C}^*, n) = \{(x_1, \dots, x_n) \in (\mathbb{C}^*)^n \mid \Gamma \cdot x_i \cap \Gamma \cdot x_j = \emptyset \text{ if } i \neq j\}$$

of ordered n -tuples of points in \mathbb{C}^* which lie in distinct orbits of the free action of $\Gamma = \mathbb{Z}/r\mathbb{Z}$ on \mathbb{C}^* by multiplication by the primitive r -th root of unity $\exp(2\pi\sqrt{-1}/r)$.

Let $B(r, n)$ denote the group with generators $\rho_0, \rho_1, \dots, \rho_{n-1}$ and relations

$$(4) \quad (\rho_0 \rho_1)^2 = (\rho_1 \rho_0)^2, \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \ (1 \leq i < n), \rho_i \rho_j = \rho_j \rho_i \ (|j-i| > 1).$$

This is the (full) monomial braid group, the fundamental group of the orbit space $M(\mathcal{A}_{r,n})/W$, where $W = G(r, n)$ is the full monomial group, cf. [BMR98]. Note that $B(r, n)$ is independent of r . This group admits a natural surjection to $G(r, n)$, which may be presented with generators $\rho_0, \rho_1, \dots, \rho_{n-1}$ and relations (4) together with $\rho_0^r = \rho_1^r = \dots = \rho_{n-1}^r = 1$. Note that the hyperplanes of $\mathcal{A}_{r,n}$ are the reflecting hyperplanes of the group $G(r, n)$, and that this group is isomorphic to the wreath product of the symmetric group Σ_n and the cyclic group $\mathbb{Z}/r\mathbb{Z}$.

The fundamental group of the complement $M(\mathcal{A}_{r,n})$ of the full monomial arrangement is the kernel $P(r, n) = \pi_1(M(\mathcal{A}_{r,n}))$ of the aforementioned surjection $B(r, n) \rightarrow G(r, n)$, which we refer to as the pure monomial braid group. Furthermore, $M(\mathcal{A}_{r,n})$ is an Eilenberg-Mac Lane space for the pure monomial braid group. A presentation for the group $P(r, n)$ is found in [Coh01, Thm. 2.2.4] (in slightly different notation).¹ For $1 \leq i \leq n$, let $X_i = \rho_{i-1} \cdots \rho_2 \rho_1 \rho_0 \rho_1 \rho_2 \cdots \rho_{i-1}$, and define

$$(5) \quad \begin{aligned} C_j &= \rho_{j-1} \cdots \rho_2 \rho_1 \rho_0^r \rho_1^{-1} \rho_2^{-1} \cdots \rho_{j-1}^{-1} \ (1 \leq j \leq n), \\ A_{i,j}^{(q)} &= X_i^{q-r} \cdot \rho_{j-1} \cdots \rho_{i+1} \rho_i^2 \rho_{i+1}^{-1} \cdots \rho_{j-1}^{-1} \cdot X_i^{r-q} \ (1 \leq i < j \leq n, \ 1 \leq q \leq r). \end{aligned}$$

These elements generate the pure monomial braid group.

Setting $r = 1$ in (3) yields a polynomial $Q_{1,n}$ which defines an arrangement $\mathcal{A}_{1,n}$ whose complement has the homotopy type of the complement of the braid arrangement \mathcal{A}_{n+1} in \mathbb{C}^{n+1} , $M(\mathcal{A}_{1,n}) \simeq M(\mathcal{A}_{n+1})$. For $r \geq 2$, the mapping $M(\mathcal{A}_{r,n}) \rightarrow M(\mathcal{A}_{1,n})$ defined by $(x_1, \dots, x_n) \mapsto (x_1^r, \dots, x_n^r)$ is a finite covering (equivalent to the pullback along the inclusion $M(\mathcal{A}_{1,n}) \hookrightarrow (\mathbb{C}^*)^n$ of the covering $(\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$ defined by the same formula). Thus, $P(r, n) = \pi_1(M(\mathcal{A}_{r,n}))$ is a finite index subgroup of $P_{n+1} = \pi_1(M(\mathcal{A}_{1,n}))$.

In this paper, building on work of Bell and Margalit [BM07] and Charney and Crisp [CC05], we find finite presentations of the automorphism groups of the pure braid groups P_n and $P(r, n)$. These automorphism groups, $\text{Aut}(P_n)$ in particular, are used in [CFR11] to study the residual freeness of these pure braid groups. The structure of the automorphism groups of the full braid groups B_n and $B(r, n)$ is known, see [DG81] and [CC05]. The monomial braid group $B(r, n) = B(2, n)$ may be realized as the Artin group of type B, and the automorphism group $\text{Aut}(B(2, n))$ was determined in [CC05] from this perspective.

¹ There is a typographical error in the second family of relations recorded in [Coh01, (2.9)]. These relations should read $[A_{j,k}^{(p)}, A_{j,l}^{[p]} A_{i,l}^{(q)} (A_{j,l}^{[p]})^{-1}]$.

2. PRELIMINARIES

In this section, we gather a number of facts regarding split extensions, (pure) braid groups, and mapping class groups which will be of use in analyzing the automorphism groups of the pure braid groups P_n and $P(r, n)$.

Let K be a group with trivial center, $Z(K) = 1$, and let A be an abelian group. As noted by Leininger and Margalit [LM06], a split central extension

$$1 \rightarrow A \rightarrow G \twoheadrightarrow K \rightarrow 1$$

induces a split extension

$$(6) \quad 1 \rightarrow \mathrm{tv}(G) \rightarrow \mathrm{Aut}(G) \twoheadrightarrow \mathrm{Aut}(K) \rightarrow 1,$$

where $\mathrm{tv}(G) < \mathrm{Aut}(G)$ is the subgroup consisting of all automorphisms of G which become trivial upon passing to the quotient K . If, moreover, $G = A \times K$ is a direct product, an explicit splitting in (6) is given by sending $\alpha \in \mathrm{Aut}(K)$ to $\tilde{\alpha} \in \mathrm{Aut}(G)$, where $\tilde{\alpha}|_A = \mathrm{id}_A$ and $\tilde{\alpha}|_K = \alpha$. We occasionally abuse notation and write simply α in place of $\tilde{\alpha}$ in this situation.

For a group G with infinite cyclic center $Z = \langle z \rangle$, a transvection is an endomorphism of G of the form $x \mapsto xz^{t(x)}$, where $t: G \rightarrow \mathbb{Z}$ is a homomorphism, see Charney and Crisp [CC05]. Such a map is an automorphism if and only if its restriction to Z is surjective, which is the case if and only if $z \mapsto z$ or $z \mapsto z^{-1}$, that is, $t(z) = 0$ or $t(z) = -2$. For the groups we are interested in, the extension $1 \rightarrow Z(G) \rightarrow G \rightarrow G/Z(G) \rightarrow 1$ is split (in fact $G \cong Z(G) \times G/Z(G)$), and $Z(G)$ is infinite cyclic. In this instance, the subgroup $\mathrm{tv}(G) < \mathrm{Aut}(G)$ consists of all transvection automorphisms of G , so we refer to $\mathrm{tv}(G)$ as the transvection subgroup of $\mathrm{Aut}(G)$.

As alluded to in the previous paragraph, the pure braid groups P_n and $P(r, n)$ admit direct product decompositions

$$(7) \quad P_n \cong Z(P_n) \times P_n/Z(P_n) \quad \text{and} \quad P(r, n) \cong Z(P(r, n)) \times P(r, n)/Z(P(r, n)),$$

and the center of each of these groups is infinite cyclic. The above direct product decompositions (the first of which, for P_n , is well known) may be obtained using results from the theory of hyperplane arrangements. See Orlik and Terao [OT92] as a general reference. First, if \mathcal{A} is a central arrangement in \mathbb{C}^n (the hyperplanes of which all contain the origin), the restriction of the Hopf bundle $\mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{CP}^{n-1}$ to the complement $M = M(\mathcal{A})$ yields a homeomorphism $M \cong \mathbb{C}^* \times \overline{M}$, where \overline{M} is the complement of the projectivization of \mathcal{A} in \mathbb{CP}^{n-1} . Thus, $\pi_1(M) \cong \mathbb{Z} \times \pi_1(\overline{M})$. Second, the braid arrangement \mathcal{A}_n and the full monomial arrangement $\mathcal{A}_{r,n}$ are fiber-type (or supersolvable) arrangements. As such, the fundamental groups of the complements decompose as iterated semidirect products of free groups,

$$P_n = \pi_1(M(\mathcal{A}_n)) = \rtimes_{k=1}^{n-1} F_k \quad \text{and} \quad P(r, n) = \pi_1(M(\mathcal{A}_{r,n})) = \rtimes_{k=1}^{n-1} F_{r(k-1)+1},$$

where F_k is the free group of rank k . The direct product decompositions (7) follow easily from these two facts. Note also that these considerations imply that the groups $\overline{P}_n = P_n/Z(P_n)$ and $\overline{P}(r, n) = P(r, n)/Z(P(r, n))$ are centerless.

Explicit generators for the centers of the braid groups B_n , P_n , $B(r, n)$, and $P(r, n)$ are known. Regarding the Artin braid groups, it is a classical result of Chow (see [Bir75, Cor. 1.8.4]) that $Z(B_n) = Z(P_n) = \mathbb{Z}$, generated by

$$Z_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n = (A_{1,2})(A_{1,3} A_{2,3}) \cdots (A_{1,n} \cdots A_{n-1,n}).$$

The centers of the monomial braid groups $B(r, n)$ and $P(r, n)$ were determined by Broué, Malle, and Rouquire [BMR98, Prop. 3.10]. In terms of the generators $\rho_0, \rho_1, \dots, \rho_{n-1}$ of $B(r, n)$, these centers are given by

$$Z(B(r, n)) = \langle (\rho_0 \rho_1 \cdots \rho_{n-1})^n \rangle \text{ and } Z(P(r, n)) = \langle (\rho_0 \rho_1 \cdots \rho_{n-1})^{nr} \rangle.$$

Write $\zeta_n = (\rho_0 \rho_1 \cdots \rho_{n-1})^n$ so that $Z(B(r, n)) = \langle \zeta_n \rangle$ and $Z(P(r, n)) = \langle \zeta_n^r \rangle$. Since $B(r, n) = B(2, n)$ is the type B Artin group, the fact that $Z(B(r, n)) = \langle \zeta_n \rangle$ follows from work of Deligne [Del72], see also Brieskorn and Saito [BS72].

We express ζ_n^r in terms of the generators of the pure monomial braid group $P(r, n)$ recorded in (5). For $1 \leq i < j \leq n$, let

$$(8) \quad \begin{aligned} A_{i,j}^{[q]} &= A_{i,j}^{(q)} A_{i,j}^{(q+1)} \cdots A_{i,j}^{(r-1)} \quad (\text{for } q < r), \\ V_{i,j}^{(q)} &= A_{i,j}^{(q)} A_{i+1,j}^{(q)} \cdots A_{j-1,j}^{(q)} \quad (\text{for } q \leq r), \\ D_k &= A_{k-1,k}^{[1]} A_{k-2,k}^{[1]} \cdots A_{1,k}^{[1]} C_k V_{1,k}^{(r)} \quad (\text{for } k \leq n). \end{aligned}$$

Lemma 1. *The center of the pure monomial braid group $P(r, n)$ is generated by*

$$\zeta_n^r = D_1 D_2 \cdots D_n.$$

Proof. Recall the braids $X_i = \rho_{i-1} \cdots \rho_2 \rho_1 \rho_0 \rho_1 \rho_2 \cdots \rho_{i-1}$ in $B(r, n)$. An inductive argument using the monomial braid relations (4) reveals that

$$\zeta_n = X_1 X_2 \cdots X_n = \zeta_{n-1} \cdot X_n.$$

The relations (4) may also be used to check that $X_i X_j = X_j X_i$ for each i and j . Thus, $Z(P(r, n))$ is generated by $\zeta_n^r = (X_1 X_2 \cdots X_n)^r = X_1^r X_2^r \cdots X_n^r = \zeta_{n-1}^r \cdot X_n^r$. We may inductively assume that $\zeta_{n-1}^r = D_1 D_2 \cdots D_{n-1}$, so it suffices to show that $D_n = X_n^r$.

Use (5) and (8) to check that $C_n V_{1,n}^{(r)} = \rho_{n-1} \cdots \rho_1 \rho_0^r \rho_1 \cdots \rho_{n-1}$ and also that $A_{i,n}^{[1]} = X_i^{1-r} (A_{i,n}^{(r)} X_i)^{r-1}$. Then, a calculation reveals that

$$D_n = (X_1 \cdots X_{n-1})^{1-r} Y_{n-1}^{r-1} \rho_{n-1} \cdots Y_1^{r-1} \rho_1 \rho_0^r \rho_1 \cdots \rho_{n-1},$$

where $Y_i = \rho_i^2 X_i$. Since $Y_i \rho_i = \rho_i X_{i+1}$ and $X_j \rho_i = \rho_i X_j$ for $i < j$, we have

$$\begin{aligned} D_n &= (X_1 \cdots X_{n-1})^{1-r} \rho_{n-1} \cdots \rho_1 X_n^{r-1} \cdots X_2^{r-1} \rho_0^r \rho_1 \cdots \rho_{n-1} \\ &= \zeta_{n-1}^{1-r} \rho_{n-1} \cdots \rho_1 \zeta_n^{r-1} \rho_0 \rho_1 \cdots \rho_{n-1} = \zeta_{n-1}^{1-r} \zeta_n^{r-1} X_n = X_n^r. \end{aligned} \quad \square$$

Recall that $\overline{P}_n = P_n / Z(P_n)$ and $\overline{P}(r, n) = P(r, n) / Z(P(r, n))$. These groups may be realized as finite index subgroups of the (extended) mapping class group of the punctured sphere. Let \mathbb{S}_m denote the sphere S^2 with m punctures, and let $\text{Mod}(\mathbb{S}_m)$ be the extended mapping class group of \mathbb{S}_m , the group of isotopy classes of all self-diffeomorphisms of \mathbb{S}_m .

The mapping class group $M(0, m)$ of isotopy classes of orientation-preserving self-diffeomorphisms of \mathbb{S}_m is an index two subgroup of $\text{Mod}(\mathbb{S}_m)$. For $m \geq 2$, the group $M(0, m)$ admits a presentation with generators $\omega_1, \dots, \omega_{m-1}$ and relations

$$(9) \quad \begin{aligned} \omega_i \omega_j &= \omega_j \omega_i \text{ for } |i - j| \geq 2, & \omega_i \omega_{i+1} \omega_i &= \omega_{i+1} \omega_i \omega_{i+1}, \\ \omega_1 \cdots \omega_{m-2} \omega_{m-1}^2 \omega_{m-2} \cdots \omega_1 &= 1, & (\omega_1 \omega_2 \cdots \omega_{m-1})^m &= 1, \end{aligned}$$

see [Bir75, Thm. 4.5]. The extended mapping class group $\text{Mod}(\mathbb{S}_m)$ then admits a presentation with the above generators and relations, along with the additional generator ϵ and relations $(\epsilon \omega_i)^2 = 1$ and $\epsilon^2 = 1$.

If G is a subgroup of a group Γ , recall that the normalizer of G in Γ is $N_\Gamma(G) = \{\gamma \in \Gamma \mid \gamma^{-1}G\gamma = G\}$, the largest subgroup of Γ having G as a normal subgroup. Building on work of Korkmaz [Kor99] and Ivanov [Iv03], Charney and Crisp [CC05, Cor. 4 (ii)] establish the following.

Proposition 2. *If $m \geq 5$ and G is a finite index subgroup of $\Gamma = \text{Mod}(\mathbb{S}_m)$, then $\text{Aut}(G) \cong N_\Gamma(G)$.*

Throughout the paper, $\text{Aut}(G)$ denotes the group of right automorphisms of G , with multiplication $\alpha \cdot \beta = \beta \circ \alpha$.

3. AUTOMORPHISMS OF THE ARTIN PURE BRAID GROUP

The map $B_n \rightarrow M(0, n+1) \rightarrow \text{Mod}(\mathbb{S}_{n+1})$ given by $\sigma_i \mapsto \omega_i$, $1 \leq i \leq n-1$, realizes $\overline{B}_n = B_n/Z$ as a finite index subgroup of the extended mapping class group $\text{Mod}(\mathbb{S}_{n+1})$, where $Z = Z(B_n) = Z(P_n)$, see, for instance, [CC05]. This comes from realizing B_n as the orientation-preserving mapping class group of \mathbb{D}_n , the n -punctured disk, relative to the boundary, and including \mathbb{D}_n in \mathbb{S}_{n+1} . In this way, $\overline{P}_n = P_n/Z$ is realized as $\text{PMod}(\mathbb{S}_{n+1})$, the subgroup of orientation-preserving mapping classes which fix every puncture.

The subgroup \overline{P}_n is normal in $\text{Mod}(\mathbb{S}_{n+1})$. Thus, $\text{Aut}(\overline{P}_n) \cong \text{Mod}(\mathbb{S}_{n+1})$ for $n \geq 4$, see Proposition 2. This fact was originally established by Korkmaz [Kor99], and extended by Bell and Margalit [BM07]. Since $P_n \cong Z \times \overline{P}_n$, the split extension (6) yields a semidirect product decomposition $\text{Aut}(P_n) \cong \text{tv}(P_n) \rtimes \text{Aut}(\overline{P}_n)$. This is an ingredient in the identification, for $n \geq 4$, of the automorphism group of the pure braid group as

$$(10) \quad \text{Aut}(P_n) \cong (\mathbb{Z}^N \rtimes \mathbb{Z}_2) \rtimes \text{Mod}(\mathbb{S}_{n+1})$$

made by Bell and Margalit [BM07, Thm. 8]. Here, $\text{tv}(P_n) \cong \mathbb{Z}^N \rtimes \mathbb{Z}_2$, where $N = \binom{n}{2} - 1$.

Recall that the center $Z = Z(P_n)$ of the pure braid group is infinite cyclic, generated by $Z_n = A_{1,2}A_{1,3}A_{2,3} \cdots A_{1,n} \cdots A_{n-1,n}$. The transvection subgroup $\text{tv}(P_n)$ of $\text{Aut}(P_n)$ consists of automorphisms of the form $A_{i,j} \mapsto A_{i,j}Z_n^{t_{i,j}}$, where $t_{i,j} \in \mathbb{Z}$ and $\sum t_{i,j}$ is either equal to 0 or -2 . In the former case, $Z_n \mapsto Z_n$, while $Z_n \mapsto Z_n^{-1}$ in the latter. This yields a surjection $\text{tv}(P_n) \rightarrow \mathbb{Z}_2$, with kernel consisting of transvections for which $\sum t_{i,j} = 0$. Since P_n has $\binom{n}{2} = N+1$ generators, this kernel is free abelian of rank N . The choice $t_{1,2} = -2$ and all other $t_{i,j} = 0$ gives a splitting $\mathbb{Z}_2 \rightarrow \text{tv}(P_n)$. Thus, $\text{tv}(P_n) \cong \mathbb{Z}^N \rtimes \mathbb{Z}_2$. This group is generated by transvections $\psi, \phi_{i,j}: P_n \rightarrow P_n$, $1 \leq i < j \leq n$, $\{i, j\} \neq \{1, 2\}$, where

$$(11) \quad \psi: A_{p,q} \mapsto \begin{cases} A_{1,2}Z_n^{-2} & p=1, q=2, \\ A_{p,q} & \text{otherwise,} \end{cases} \quad \phi_{i,j}: A_{p,q} \mapsto \begin{cases} A_{1,2}Z_n & p=1, q=2, \\ A_{i,j}Z_n^{-1} & p=i, q=j, \\ A_{p,q} & \text{otherwise.} \end{cases}$$

It is readily checked that $\psi^2 = 1$ and that $\psi\phi_{i,j}\psi = \phi_{i,j}^{-1}$. Observe that nontrivial elements of $\text{tv}(P_n)$ are outer automorphisms.

The mapping class group $\text{Mod}(\mathbb{S}_{n+1})$ acts on $\overline{P}_n = P_n/\langle Z_n \rangle \cong \text{PMod}(\mathbb{S}_{n+1})$ by conjugation. We exhibit automorphisms of P_n which fix the generator Z_n of the center and induce the corresponding automorphisms of \overline{P}_n upon passing to the quotient. For group elements x and y , write $y^x = x^{-1}yx$.

Define elements ω_k , $1 \leq k \leq n$, and ϵ of $\text{Aut}(P_n)$ as follows:

$$(12) \quad \begin{aligned} \omega_k : A_{i,j} &\mapsto \begin{cases} A_{i-1,j} & \text{if } k = i - 1, \\ A_{i+1,j}^{A_{i,i+1}} & \text{if } k = i < j - 1, \\ A_{i,j-1} & \text{if } k = j - 1 > i, \\ A_{i,j+1}^{A_{i,j+1}} & \text{if } k = j, \\ A_{i,j} & \text{otherwise,} \end{cases} \quad \text{for } 1 \leq k \leq n - 1, k \neq 2, \\ \omega_2 : A_{i,j} &\mapsto \begin{cases} A_{1,3}^{A_{2,3}} Z_n & \text{if } i = 1, j = 2, \\ A_{1,2} Z_n^{-1} & \text{if } i = 1, j = 3, \\ A_{3,j}^{A_{2,3}} & \text{if } i = 2, j \geq 4, \\ A_{2,j} & \text{if } i = 3, \\ A_{i,j} & \text{otherwise,} \end{cases} \\ \omega_n : A_{i,j} &\mapsto \begin{cases} A_{i,j} & \text{if } j \neq n, \\ (A_{1,n} A_{1,2} A_{1,3} \cdots A_{1,n-1})^{-1} Z_n & \text{if } i = 1, j = n, \\ (A_{2,n} A_{1,2} A_{2,3} \cdots A_{2,n-1})^{-1} Z_n & \text{if } i = 2, j = n, \\ (A_{i,n} A_{1,i} \cdots A_{i-1,i} A_{i,i+1} \cdots A_{i,n-1})^{-1} & \text{if } 3 \leq i, j = n, \end{cases} \\ \epsilon : A_{i,j} &\mapsto \begin{cases} A_{1,2}^{-1} Z_n^2 & \text{if } i = 1, j = 2, \\ (A_{i+1,j} \cdots A_{j-1,j})^{-1} A_{i,j}^{-1} (A_{i+1,j} \cdots A_{j-1,j}) & \text{otherwise.} \end{cases} \end{aligned}$$

Check that $\omega_k(Z_n) = Z_n$ for each k , and $\epsilon(Z_n) = Z_n$. Also, note that, for $1 \leq k \leq n - 1$ and $k \neq 2$, the automorphism ω_k is given by the usual conjugation action of the braid σ_k on the pure braid group, $\omega_k(A_{i,j}) = A_{i,j}^{\sigma_k} = \sigma_k^{-1} A_{i,j} \sigma_k$, see [DG81]. The automorphism ω_2 is the composite of the conjugation action of σ_2 and the transvection $\phi_{1,3}$, see (11). This accounts for the fact that $A_{1,2} = [(A_{1,3} A_{2,3}) \cdots (A_{1,n} \cdots A_{n-1,n})]^{-1}$ in \overline{P}_n , the fact that, for instance, $A_{1,3}^{\sigma_2} = A_{1,2}$ in P_n , and insures that $\omega_2(Z_n) = Z_n$.

Similar considerations explain the occurrence of Z_n in the formulas for the automorphisms ω_n and ϵ above. The former automorphism of P_n lifts the automorphism of \overline{P}_n given by conjugation by $\omega_n \in \text{Mod}(\mathbb{S}_{n+1})$. This conjugation action can be determined using the mapping class group relations (9), noting that the relations $\omega_i \omega_{i+1} \omega_i = \omega_{i+1} \omega_i \omega_{i+1}$ and $\omega_1 \cdots \omega_{n-1} \omega_n^2 \omega_{n-1} \cdots \omega_1 = 1$ imply that, for instance,

$$\begin{aligned} \omega_n^{-1} A_{n-1,n} \omega_n &= \omega_n^{-1} \omega_{n-1}^2 \omega_n = \omega_{n-1} \omega_n^2 \omega_{n-1}^{-1} \\ &= \omega_{n-1} (\omega_{n-1} \cdots \omega_2 \omega_1^2 \omega_2 \cdots \omega_{n-1})^{-1} \omega_{n-1}^{-1} \\ &= \omega_{n-1} (A_{1,n} A_{2,n} \cdots A_{n-1,n})^{-1} \omega_{n-1}^{-1} \\ &= (A_{n-1,n} A_{1,n-1} \cdots A_{n-2,n-1})^{-1}. \end{aligned}$$

Similarly, the fact that $A_{i,n} = A_{n-1,n}^{\omega_{n-2} \cdots \omega_i}$ for $i \leq n - 2$ may be used to calculate $\omega_n^{-1} A_{i,n} \omega_n$.

Proposition 3. *The elements $\omega_1, \dots, \omega_n, \epsilon \in \text{Aut}(P_n)$ satisfy the mapping class group relations (9) and the relations $\epsilon^2 = 1$ and $(\epsilon \omega_k)^2 = 1$ for each k , $1 \leq k \leq n$.*

Proof. As noted above, for $1 \leq k \leq n - 1$ and $k \neq 2$, the automorphism ω_k is given by the conjugation action of the braid σ_k , $\omega_k(A_{i,j}) = A_{i,j}^{\sigma_k}$. It follows that all of the (braid) relations (9) that do not involve ω_2 or ω_n hold. So it remains

to check that the automorphisms ω_k of P_n satisfy $\omega_i\omega_2\omega_i = \omega_2\omega_i\omega_2$ for $i = 1, 3$, $\omega_2\omega_i = \omega_i\omega_2$ for $i \geq 4$, $\omega_{n-1}\omega_n\omega_{n-1} = \omega_n\omega_{n-1}\omega_n$, $\omega_i\omega_n = \omega_n\omega_i$ for $i \leq n-2$, $\omega_1 \cdots \omega_{n-1}\omega_n^2\omega_{n-1} \cdots \omega_1 = 1$, and $(\omega_1\omega_2 \cdots \omega_n)^{n+1} = 1$. We will check the last two, and leave the others as exercises for the reader.

To verify that $\omega_1 \cdots \omega_{n-1}\omega_n^2\omega_{n-1} \cdots \omega_1 = 1$, first check that

$$\begin{aligned} \omega_1 \cdots \omega_{n-1}(A_{i,j}) &= \begin{cases} Z_n A_{1,n}^{A_{2,n} \cdots A_{n-1,n}} & \text{if } i = 1, j = 2, \\ Z_n^{-1} A_{1,2} & \text{if } i = 2, j = 3, \\ A_{j-1,n}^{A_{j,n} \cdots A_{n-1,n} (A_{1,j-1} \cdots A_{j-2,j-1})^{-1}} & \text{if } i = 1, j \geq 3, \\ A_{i-1,j-1} & \text{otherwise,} \end{cases} \\ \omega_n^2(A_{i,j}) &= \begin{cases} A_{i,j} & \text{if } j \leq n-1, \\ A_{i,n}^{A_{1,i} \cdots A_{i-1,i} A_{i,i+1} \cdots A_{i,n-1}} & \text{if } j = n, \end{cases} \\ \omega_{n-1} \cdots \omega_1(A_{i,j}) &= \begin{cases} A_{2,3} Z_n & \text{if } i = 1, j = 2, \\ A_{1,2} Z_n^{-1} & \text{if } i = 1, j = n, \\ A_{1,i+1} & \text{if } i \geq 2, j = n, \\ A_{i+1,j+1} & \text{otherwise.} \end{cases} \end{aligned}$$

These calculations, together with the pure braid relations (2), can be used to check that $\omega_1 \cdots \omega_{n-1}\omega_n^2\omega_{n-1} \cdots \omega_1 = 1$.

To verify that $(\omega_1\omega_2 \cdots \omega_n)^{n+1} = 1$, first note that the relations $\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1}$ for $1 \leq i \leq n$ and $\omega_j\omega_i = \omega_i\omega_j$ for $|j-i| \geq 2$ imply that

$$(\omega_1\omega_2 \cdots \omega_n)^{n+1} = (\omega_1\omega_2 \cdots \omega_{n-1})^n \cdot \omega_n \cdots \omega_2\omega_1^2\omega_2 \cdots \omega_n, \text{ and}$$

$$\omega_n \cdots \omega_2\omega_1^2\omega_2 \cdots \omega_n = (\omega_n \cdots \omega_1) \cdot \omega_1 \cdots \omega_{n-1}\omega_n^2\omega_{n-1} \cdots \omega_1 (\omega_n \cdots \omega_1)^{-1}.$$

Since $\omega_1 \cdots \omega_{n-1}\omega_n^2\omega_{n-1} \cdots \omega_1 = 1$ by the previous paragraph, it suffices to check that $(\omega_1\omega_2 \cdots \omega_{n-1})^n = 1$.

Write $\tau = \omega_1 \cdots \omega_{n-1}$. We must show that $\tau^n = 1$. The action of τ on the pure braid generators $A_{i,j}$ is given above. In particular, $\tau(A_{i,j}) = A_{i-1,j-1}$ for $i \geq 2$ and $j \geq 4$. Also, note that for $j \geq 3$, the pure braid relations (2) may be used to show that

$$\tau(A_{1,j}) = A_{j-1,n}^{A_{j,n} \cdots A_{n-1,n} (A_{1,j-1} \cdots A_{j-2,j-1})^{-1}} = A_{j-1,n}^{A_{1,n} \cdots A_{n-1,n}}.$$

Observe that $\tau^{n-3}(A_{n-1,n}) = A_{2,3}$. Consequently, $\tau^{n-2}(A_{n-1,n}) = Z_n^{-1}A_{1,2}$, and $\tau^{n-1}(A_{n-1,n}) = A_{1,n}^{A_{2,n} \cdots A_{n-1,n}}$. A calculation then reveals that $\tau^n(A_{n-1,n}) = A_{n-1,n}$. It follows that $\tau^{n-k}(A_{k-1,k}) = A_{n-1,n}$ for $k \geq 3$, which implies that $\tau^n(A_{k-1,k}) = A_{k-1,k}$ for $k \geq 3$.

If $i = j - k$ with $k \geq 2$ (so that $j \geq 3$), then $A_{i,j} = A_{j-k,j} = \tau^{n-j}(A_{n-k,n})$. If $\tau^j(A_{i,j}) = A_{n-k,n}$, it follows that $\tau^n(A_{n-k,n}) = A_{n-k,n}$ and then that $\tau^n(A_{i,j}) = A_{i,j}$. Thus, it suffices to show that $\tau^j(A_{i,j}) = A_{n-k,n}$. If $i > 1$, then $\tau^{i-1}(A_{i,j}) = A_{1,j-i+1}$. So it is enough to show that $\tau^q(A_{1,q}) = A_{n-q+1,n}$, where $q \geq 3$. Checking that

$$\tau^p(A_{1,q}) = A_{q-p,n-p+1}^{A_{1,n-p+1} \cdots A_{n-p,n-p+1} \cdots A_{n-p+1,n-p+2} \cdots A_{n-p+1,n}}$$

for $1 \leq p \leq q-1$, we have

$$\begin{aligned} \tau^q(A_{1,q}) &= \tau(A_{1,n-q}^{A_{2,n-q} \cdots A_{n-q-1,n-q} \cdots A_{n-q,n-q+1} \cdots A_{n-q,n}}) \\ &= A_{n-q-1,n}^{A_{n-q,n} \cdots A_{n-1,n} \cdots A_{n-q-1,n-q} \cdots A_{n-q-1,n}} \end{aligned}$$

A calculation with the pure braid relations (2) then shows that $\tau^q(A_{1,q}) = A_{n-q+1,n}$.

It remains to check that $\epsilon^2 = 1$ and $(\epsilon\omega_k)^2 = 1$ for each k , $1 \leq k \leq n$. The first of these is straightforward. For the remaining ones, note that $\epsilon(A_{i,j} \cdots A_{j-1,j}) = (A_{i,j} \cdots A_{j-1,j})^{-1}$ for $i > 1$ and $j > 2$, $\epsilon(A_{i,i+1} \cdots A_{i,j}) = (A_{i,i+1} \cdots A_{i,j})^{-1}$ for $i > 1$, while $\epsilon(A_{1,2} \cdots A_{1,j}) = (A_{1,2} \cdots A_{1,j})^{-1} Z_n^2$. These observations, together with the pure braid relations (2) may be used to verify that $\omega_k \epsilon \omega_k = \epsilon$ for each k , $1 \leq k \leq n$. \square

Thus, the elements $\omega_1, \dots, \omega_n$ and ϵ of $\text{Aut}(P_n)$ satisfy the relations of the extended mapping class group $\text{Mod}(\mathbb{S}_{n+1})$. By construction, these elements of $\text{Aut}(P_n)$ induce the automorphisms of $\overline{P}_n \cong \text{PMod}(\mathbb{S}_{n+1})$ corresponding to conjugation by the generators (with the same names) of $\text{Mod}(\mathbb{S}_{n+1})$ upon passing to the quotient.

Theorem 4. *For $n \geq 4$, the automorphism group $\text{Aut}(P_n)$ of the pure braid group admits a presentation with generators*

$$\epsilon, \omega_k, 1 \leq k \leq n, \psi, \phi_{i,j}, 1 \leq i < j \leq n, \{i,j\} \neq \{1,2\},$$

and relations

$$\begin{aligned} \omega_i \omega_j &= \omega_j \omega_i, \quad |i-j| \geq 2, \quad \omega_i \omega_{i+1} \omega_i = \omega_{i+1} \omega_i \omega_{i+1}, \quad i < n, \quad \epsilon^2 = 1, \\ (\omega_1 \omega_2 \cdots \omega_n)^{n+1} &= 1, \quad \omega_1 \cdots \omega_{n-1} \omega_n^2 \omega_{n-1} \cdots \omega_1 = 1, \quad (\epsilon \omega_k)^2 = 1, \quad k \leq n, \\ \psi \phi_{i,j} \psi &= \phi_{i,j}^{-1}, \quad \forall i, j, \quad \phi_{i,j} \phi_{p,q} = \phi_{p,q} \phi_{i,j}, \quad \forall i, j, p, q, \quad \psi^2 = 1, \\ \epsilon \psi \epsilon &= \psi, \quad \omega_k^{-1} \psi \omega_k = \psi, \quad k \leq n, \quad \epsilon \phi_{i,j} \epsilon = \phi_{i,j}^{-1}, \\ \omega_1^{-1} \phi_{i,j} \omega_1 &= \begin{cases} \phi_{2,j} & i = 1, \\ \phi_{1,j} & i = 2, \\ \phi_{i,j} & \text{otherwise,} \end{cases} \quad \omega_2^{-1} \phi_{i,j} \omega_2 = \begin{cases} \phi_{1,3}^{-1} & i = 1, j = 3, \\ \phi_{1,3}^{-1} \phi_{3,j} & i = 2, j > 3, \\ \phi_{1,3}^{-1} \phi_{2,j} & i = 3, \\ \phi_{1,3}^{-1} \phi_{i,j} & \text{otherwise,} \end{cases} \\ \omega_k^{-1} \phi_{i,j} \omega_k &= \begin{cases} \phi_{i-1,j} & k = i-1, \\ \phi_{i+1,j} & k = i < j-1, \\ \phi_{i,j-1} & k = j-1 > i, \quad \text{for } 3 \leq k \leq n-1, \\ \phi_{i,j+1} & k = j, \\ \phi_{i,j} & \text{otherwise,} \end{cases} \\ \omega_n^{-1} \phi_{i,j} \omega_n &= \begin{cases} \phi_{i,j} \phi_{1,n} \phi_{2,n} \phi_{i,n}^{-1} \phi_{j,n}^{-1} & j < n, \\ \phi_{i,n}^{-1} \phi_{1,n} \phi_{2,n} & j = n. \end{cases} \end{aligned}$$

Proof. Recall from (6) and (10) that there is a split, short exact sequence

$$1 \rightarrow \text{tv}(P_n) \rightarrow \text{Aut}(P_n) \hookrightarrow \text{Mod}(\mathbb{S}_{n+1}) \rightarrow 1.$$

Since the automorphisms ψ and $\phi_{i,j}$ generate the transvection subgroup $\text{tv}(P_n)$, and the automorphisms ϵ and ω_k induce the generators of $\text{Mod}(\mathbb{S}_{n+1}) = \text{Aut}(\overline{P}_n)$, these automorphisms collectively generate $\text{Aut}(P_n)$. By Proposition 3, the automorphisms ϵ and ω_k satisfy the extended mapping class group relations. As noted previously, the formulas (11) may be used to show that the transvections ψ and $\phi_{i,j}$ satisfy $\psi^2 = 1$ and $\psi \phi_{i,j} \psi = \phi_{i,j}^{-1}$. So it suffices to show that the actions of the automorphisms ϵ and ω_k on the transvections ψ and $\phi_{i,j}$ are as asserted. This may be

accomplished by calculations with the explicit descriptions of these automorphisms given in (11) and (12). \square

Remark 5. Theorem 4 exhibits the semidirect product structure of $\text{Aut}(P_n) \cong \text{tv}(P_n) \rtimes \text{Mod}(\mathbb{S}_{n+1})$. Recall that $\text{Mod}(\mathbb{S}_{n+1}) = M(0, n+1) \rtimes \mathbb{Z}_2$ is itself the semidirect product of the (non-extended) mapping class group and \mathbb{Z}_2 . Note that the generator ψ of $\text{tv}(P_n) < \text{Aut}(P_n)$ commutes with the generators $\epsilon, \omega_1, \dots, \omega_n$ of $\text{Aut}(P_n)$ which induce the generators of $\text{Mod}(\mathbb{S}_{n+1})$. It follows that $\text{Aut}(P_n)$ may be realized as the iterated semidirect product $\text{Aut}(P_n) \cong (\mathbb{Z}^N \rtimes M(0, n+1)) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Similar considerations yield a presentation for the automorphism group of the three strand pure braid group $P_3 \cong \mathbb{Z} \times F_2$. In this case, the split extension (6) yields $\text{Aut}(P_3) \cong \text{tv}(P_3) \rtimes \text{Aut}(F_2)$, where $\text{tv}(P_3) \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_2$, generated by $\psi, \phi_{1,3}, \phi_{2,3}$ with $\psi^2 = 1$ and $\psi\phi_{i,j}\psi = \phi_{i,j}^{-1}$ (see (11)), and

$$F_2 = \overline{P}_3 = P_3/Z(P_3) = P_3/\langle A_{1,2}A_{1,3}A_{2,3} \rangle = \langle A_{1,3}, A_{2,3} \rangle$$

is the free group on two generators. The group $\text{Aut}(F_2)$ admits the following presentation, due to Neumann (see [MKS66, §3.5, Prob. 2]):

$$\text{Aut}(F_2) = \langle P, \sigma, U \mid P^2, \sigma^2, (\sigma P)^4, (P\sigma PU)^2, (UP\sigma)^3, [U, \sigma U\sigma] \rangle,$$

where the automorphisms P, σ, U of F_2 are given by

$$\begin{aligned} P(A_{1,3}) &= A_{2,3}, & \sigma(A_{1,3}) &= A_{1,3}^{-1}, & U(A_{1,3}) &= A_{1,3}A_{2,3}, \\ P(A_{2,3}) &= A_{1,3}, & \sigma(A_{2,3}) &= A_{2,3}, & U(A_{2,3}) &= A_{2,3}. \end{aligned}$$

Lifts of these automorphisms to automorphisms of P_3 fixing $Z_3 = A_{1,2}A_{1,3}A_{2,3}$ are given by setting

$$P(A_{1,2}) = A_{2,3}A_{1,2}A_{2,3}^{-1}, \quad \sigma(A_{1,2}) = A_{1,2}A_{1,3}^2, \quad U(A_{1,2}) = A_{2,3}^{-1}A_{1,2}.$$

Calculations with these formulas yield the following result.

Proposition 6. *The automorphism group $\text{Aut}(P_3)$ of the three strand pure braid group admits a presentation with generators $P, \sigma, U, \psi, \phi_{1,3}, \phi_{2,3}$, and relations*

$$\begin{aligned} &P^2, \quad \sigma^2, \quad (\sigma P)^4, \quad (P\sigma PU)^2, \quad (UP\sigma)^3, \quad [U, \sigma U\sigma], \\ &[U, \psi], \quad [P, \psi], \quad [\sigma, \psi], \quad [U, \phi_{1,3}], \quad P\phi_{1,3}P\phi_{2,3}^{-1}, \quad (\sigma\phi_{1,3})^2, \\ &\psi^2, \quad (\psi\phi_{i,3})^2, \quad [\phi_{1,3}, \phi_{2,3}], \quad \phi_{1,3}[\phi_{2,3}, U], \quad P\phi_{2,3}P\phi_{1,3}^{-1}, \quad [\sigma, \phi_{2,3}]. \end{aligned}$$

Remark 7. Note that the generator ψ of $\text{tv}(P_3) < \text{Aut}(P_3)$ commutes with the generators P, σ, U of $\text{Aut}(P_3)$ which project to the generators of $\text{Aut}(F_2)$. It follows that $\text{Aut}(P_3) \cong \mathbb{Z}^2 \rtimes (\mathbb{Z}_2 \times \text{Aut}(F_2))$.

Since the two strand pure braid group $P_2 = \mathbb{Z}$ is infinite cyclic, $\text{Aut}(P_2) = \mathbb{Z}_2$.

4. AUTOMORPHISMS OF THE PURE MONOMIAL BRAID GROUP

As discussed for example in [BMR98, §3], the full monomial braid group $B(r, n) = B(2, n)$ embeds in the Artin braid group B_{n+1} . In terms of the standard generators $\sigma_i, 1 \leq i \leq n$, of B_{n+1} and the generators $\rho_j, 0 \leq j \leq n-1$ of $B(r, n)$, one choice of embedding is given by $\rho_0 \mapsto \sigma_1^2$ and $\rho_j \mapsto \sigma_{j+1}$ for $j \neq 0$. Restricting to the pure monomial braid group yields a monomorphism $P(r, n) \rightarrow P_{n+1}$. In terms of the generators (1) of P_{n+1} and (5) of $P(r, n)$, this is given by

$$C_j \mapsto A_{1,j+1}^r, \quad A_{i,j}^{(q)} \mapsto (A_{1,i+1} \cdots A_{i,i+1})^{q-r} A_{i+1,j+1} (A_{1,i+1} \cdots A_{i,i+1})^{r-q}.$$

Recall the generators $Z_{n+1} = (\sigma_1 \cdots \sigma_n)^{n+1}$ and $\zeta_n = (\rho_0 \cdots \rho_{n-1})^n$ of the centers $Z(B_{n+1}) = Z(P_{n+1})$ and $Z(B(r, n))$, and that $Z(P(r, n))$ is generated by ζ_n^r . It is readily checked that the above embedding takes ζ_n to Z_{n+1} . Consequently, the group $\overline{P}(r, n) = P(r, n)/Z(P(r, n))$ may be realized as a (finite index) subgroup of $\overline{P}_{n+1} = P_{n+1}/Z(P_{n+1})$.

Composing with the map $B_{n+1} \rightarrow \text{Mod}(\mathbb{S}_{n+2})$ given by $\sigma_i \mapsto \omega_i$ realizes $\overline{P}(r, n)$ as a finite index subgroup of the extended mapping class group $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$. Hence, for $n \geq 3$, we have $\text{Aut}(\overline{P}(r, n)) \cong N_\Gamma(\overline{P}(r, n))$ by Proposition 2. Since $P(r, n) \cong Z(P(r, n)) \times \overline{P}(r, n)$, the split extension (6) yields a semidirect product decomposition $\text{Aut}(P(r, n)) \cong \text{tv}(P(r, n)) \rtimes \text{Aut}(\overline{P}(r, n))$. Thus, for $n \geq 3$, the automorphism group of the pure monomial braid group may be realized as

$$\text{Aut}(P(r, n)) \cong \text{tv}(P(r, n)) \rtimes N_\Gamma(\overline{P}(r, n)).$$

Lemma 8. *Let $N_r = r \binom{n}{2} + n - 1$. The transvection subgroup of the automorphism group of the pure monomial braid group is given by $\text{tv}(P(r, n)) \cong \mathbb{Z}^{N_r} \rtimes \mathbb{Z}_2$, where \mathbb{Z}_2 acts on \mathbb{Z}^{N_r} by taking elements to their inverses.*

Proof. For notational convenience, denote the generator of the center of $P(r, n)$ by $Z_{r,n} = \zeta_n^r$. In terms of the generators (5) of $P(r, n)$, the transvection subgroup $\text{tv}(P(r, n))$ of $\text{Aut}(P(r, n))$ consists of automorphisms of the form $C_j \mapsto C_j Z_{r,n}^{s_j}$ and $A_{i,j}^{(q)} \mapsto A_{i,j}^{(q)} Z_{r,n}^{t_{i,j,q}}$, where $s_j, t_{i,j,q} \in \mathbb{Z}$ and $S = \sum_{j=1}^n s_j + \sum_{q=1}^r \sum_{1 \leq i < j \leq n} t_{i,j,q}$ is either equal to 0 or -2 . In the former case, $Z_{r,n} \mapsto Z_{r,n}$, while $Z_{r,n} \mapsto Z_{r,n}^{-1}$ in the latter. This yields a surjection $\text{tv}(P(r, n)) \rightarrow \mathbb{Z}_2$, with kernel consisting of transvections for which $S = 0$. Since $P(r, n)$ has $N_r + 1$ generators, this kernel is free abelian of rank N_r . Setting $s_1 = -2$, $s_j = 0$ for $2 \leq j \leq n$, and all $t_{i,j,q} = 0$ gives a splitting $\mathbb{Z}_2 \rightarrow \text{tv}(P(r, n))$. Thus, $\text{tv}(P(r, n)) \cong \mathbb{Z}^{N_r} \rtimes \mathbb{Z}_2$. This group is generated by transvections Ψ , Υ_i , $2 \leq i \leq n$, $\Phi_{i,j,p}$, $1 \leq i < j \leq n$, $1 \leq p \leq r$, of $P(r, n)$, defined by

$$(13) \quad \begin{aligned} \Psi: & \begin{cases} C_j \mapsto C_1 Z_{r,n}^{-2} & \text{if } j = 1, \\ C_j \mapsto C_j & \text{if } j \neq 1, \\ A_{k,l}^{(q)} \mapsto A_{k,l}^{(q)} & \text{for all } k, l, q, \end{cases} & \Upsilon_i: & \begin{cases} C_j \mapsto C_1 Z_{r,n} & \text{if } j = 1, \\ C_j \mapsto C_i Z_{r,n}^{-1} & \text{if } j = i, \\ C_j \mapsto C_j, & \text{if } j \neq 1, i, \\ A_{k,l}^{(q)} \mapsto A_{k,l}^{(q)} & \text{for all } k, l, q, \end{cases} \\ \Phi_{i,j,p}: & \begin{cases} C_j \mapsto C_1 Z_{r,n} & \text{if } j = 1, \\ C_j \mapsto C_j & \text{if } j \neq 1, \\ A_{k,l}^{(q)} \mapsto A_{i,j}^{(p)} Z_{r,n}^{-1} & \text{if } k = i, l = j, q = p, \\ A_{k,l}^{(q)} \mapsto A_{k,l}^{(q)} & \text{otherwise.} \end{cases} \end{aligned}$$

Check that the transvections Υ_i , $\Phi_{i,j,p}$ all commute, and that $\Psi^2 = 1$, $\Psi \Upsilon_i \Psi = \Upsilon_i^{-1}$, and $\Psi \Phi_{i,j,p} \Psi = \Phi_{i,j,p}^{-1}$ to complete the proof. \square

For $n \geq 3$, viewing the group $\overline{P}(r, n)$ as a subgroup of the extended mapping class group via the sequence of embeddings

$$\overline{P}(r, n) \rightarrow \overline{P}_{n+1} \rightarrow \overline{B}_{n+1} \rightarrow M(0, n+2) \rightarrow \text{Mod}(\mathbb{S}_{n+2}),$$

the group $\text{Mod}(\mathbb{S}_{n+2})$ acts on $\overline{P}(r, n)$ by conjugation. The subgroup $\overline{P}(r, n) < \text{Mod}(\mathbb{S}_{n+2})$ is, however, not a normal subgroup. For instance, one can check that

$\omega_1 \cdot \overline{P}(r, n) \neq \overline{P}(r, n) \cdot \omega_1$. Thus, the normalizer $N_\Gamma(\overline{P}(r, n))$ of $\overline{P}(r, n)$ in $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$ is a proper subgroup of $\text{Mod}(\mathbb{S}_{n+2})$.

So to understand the structure of $\text{Aut}(P(r, n)) = \text{tv}(P(r, n)) \rtimes N_\Gamma(\overline{P}(r, n))$, we must determine this normalizer. For $n \geq 3$, the normalizer $N_\Gamma(\overline{B}(r, n))$ of $\overline{B}(r, n) = \overline{B}(2, n) = B(2, n)/Z(B(2, n))$ in $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$ was found by Charney and Crisp [CC05, Prop. 10]:

$$(14) \quad N_\Gamma(\overline{B}(r, n)) \cong \overline{B}(r, n) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

Identifying the generators of $\overline{B}(r, n)$ with their images in $\text{Mod}(\mathbb{S}_{n+2})$, the group $N_\Gamma(\overline{B}(r, n))$ has generators $\rho_0 = \omega_1^2, \rho_1 = \omega_2, \dots, \rho_{n-1} = \omega_n, \epsilon, \Delta$, where

$$\Delta = \omega_1 \cdots \omega_{n+1} \cdot \omega_1 \cdots \omega_n \cdot \omega_1 \cdots \omega_{n-1} \cdots \omega_1 \cdot \omega_2 \cdot \omega_1$$

in $\text{Mod}(\mathbb{S}_{n+2})$. Note that $\Delta^2 = (\omega_1 \cdots \omega_{n+1})^{n+2} = 1$. The elements ϵ and Δ generate $\mathbb{Z}_2 \times \mathbb{Z}_2$. Their action on $\overline{B}(r, n)$ is given by $\epsilon: \rho_i \mapsto \rho_i^{-1}$ and

$$\Delta: \rho_i \mapsto \begin{cases} (\rho_{n-1} \cdots \rho_1 \rho_0 \rho_1 \cdots \rho_{n-1})^{-1} & \text{if } i = 0, \\ \rho_{n-i} & \text{if } 1 \leq i \leq n-1. \end{cases}$$

Proposition 9. *Let $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$. For $n \geq 3$, $N_\Gamma(\overline{P}(r, n)) = N_\Gamma(\overline{B}(r, n))$.*

Proof. Since $\overline{P}(r, n)$ is normal in $\overline{B}(r, n)$, we have $\rho_i(\overline{P}(r, n)) = \overline{P}(r, n)$ for each i , $0 \leq i \leq n-1$. It is straightforward to check that $\epsilon(\overline{P}(r, n)) = \overline{P}(r, n)$. We assert that $\Delta(\overline{P}(r, n)) = \overline{P}(r, n)$ as well, which would imply that $\overline{P}(r, n)$ is normal in $N_\Gamma(\overline{B}(r, n))$.

For this, recall the monomial braids $X_i = \rho_{i-1} \cdots \rho_1 \rho_0 \rho_1 \cdots \rho_{i-1}$, and note that $\Delta(\rho_0) = X_n^{-1}$ and more generally, $\Delta(X_i) = X_{n-i+1}^{-1}$. Recall also from the proof of Lemma 1 that $X_i^r = D_i$ is a pure monomial braid. Using these observations, one can check (on the generators of $\overline{P}(r, n)$, see (5)) that $\Delta(\overline{P}(r, n)) = \overline{P}(r, n)$. Thus, $\overline{P}(r, n) \triangleleft N_\Gamma(\overline{B}(r, n))$.

The above considerations imply that $N_\Gamma(\overline{B}(r, n))$ is a subgroup of $N_\Gamma(\overline{P}(r, n))$, since the latter is the largest subgroup of $\text{Mod}(\mathbb{S}_{n+2})$ in which $\overline{P}(r, n)$ is normal. However, the (right) cosets of $H = N_\Gamma(\overline{B}(r, n))$ in $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$ are H and $H \cdot \omega_1$, and since $\omega_1 \cdot \overline{P}(r, n) \neq \overline{P}(r, n) \cdot \omega_1$, the same is true for any element of $H \cdot \omega_1$. It follows that $N_\Gamma(\overline{P}(r, n)) = N_\Gamma(\overline{B}(r, n))$. \square

Hence we have $\text{Aut}(P(r, n)) \cong \text{tv}(P(r, n)) \rtimes N_\Gamma(\overline{B}(r, n))$, and we now turn our attention to exhibiting a presentation for this group. As done with the Artin pure braid group in the previous section, we exhibit automorphisms of $P(r, n)$ which fix the generator $Z_{r,n} = \zeta_n^r$ of the center, and induce the corresponding (conjugation) automorphisms of $\overline{P}(r, n)$ upon passing to the quotient.

The automorphisms ϵ and Δ of $\overline{B}(r, n)$ extend to automorphisms of $B(r, n)$ (denoted by the same symbols) which take the generator ζ_n of the center $Z(B(r, n))$ to its inverse. For $\beta \in B(r, n)$, let $c_\beta \in \text{Aut}(P(r, n))$ be the automorphism given by conjugation by β , $c_\beta(x) = \beta^{-1}x\beta$. Recall the transvection automorphisms Ψ , Υ_i , $\Phi_{i,j,p}$ of $P(r, n)$ defined in (13), and define elements $\tilde{\rho}_k$, $0 \leq k \leq n-1$, $\tilde{\epsilon}$, and $\tilde{\Delta}$ of $\text{Aut}(P(r, n))$ as follows:

$$(15) \quad \tilde{\rho}_0 = c_{\rho_0}, \quad \tilde{\rho}_1 = c_{\rho_1} \circ \Upsilon_2, \quad \tilde{\rho}_k = c_{\rho_k} \quad (2 \leq k \leq n-1), \quad \tilde{\epsilon} = \epsilon \circ \Psi, \quad \tilde{\Delta} = \Delta \circ \Psi \circ \Upsilon_n.$$

Since $c_\beta(Z_{r,n}) = Z_{r,n}$, $\epsilon(Z_{r,n}) = Z_{r,n}^{-1}$, $\Delta(Z_{r,n}) = Z_{r,n}^{-1}$, $\Upsilon_j(Z_{r,n}) = Z_{r,n}$, and $\Psi(Z_{r,n}) = Z_{r,n}^{-1}$, each of the automorphisms defined above fixes $Z_{r,n}$. Explicit formulas for the actions of these automorphisms on the pure monomial braid generators (5) may be obtained through calculations using the monomial braid relations (4) and the presentation for $P(r, n)$ found in [Coh01, Thm. 2.2.4] (see also [Coh01, Lem. 2.2.3]). The results of these calculations are relegated to the next section.

Proposition 10. *The automorphisms $\tilde{\rho}_0, \dots, \tilde{\rho}_{n-1}, \tilde{\epsilon}, \tilde{\Delta} \in \text{Aut}(P(r, n))$ satisfy*

$$\begin{aligned} \tilde{\rho}_i \tilde{\rho}_{i+1} \tilde{\rho}_i &= \tilde{\rho}_{i+1} \tilde{\rho}_i \tilde{\rho}_{i+1} \text{ for } 1 \leq i < n, & \tilde{\rho}_i \tilde{\rho}_j &= \tilde{\rho}_j \tilde{\rho}_i \text{ for } |i - j| \geq 2, & \tilde{\epsilon}^2 &= 1, \\ (\tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1})^n &= 1, & (\tilde{\rho}_0 \tilde{\rho}_1)^2 &= (\tilde{\rho}_1 \tilde{\rho}_0)^2, & \tilde{\Delta}^2 &= 1, \\ \tilde{\Delta} \tilde{\rho}_0 \tilde{\Delta} &= (\tilde{\rho}_{n-1} \cdots \tilde{\rho}_1 \tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1})^{-1}, & (\tilde{\epsilon} \tilde{\rho}_k)^2 &= 1 \text{ for } 0 \leq k < n, & [\tilde{\epsilon}, \tilde{\Delta}] &= 1, \\ \tilde{\Delta} \tilde{\rho}_k \tilde{\Delta} &= \tilde{\rho}_{n-k} \text{ for } 1 \leq k < n. \end{aligned}$$

These are the relations of the normalizer of $\overline{P}(r, n)$ in $\text{Mod}(\mathbb{S}_{n+2})$.

Sketch of proof. Since $\tilde{\rho}_k$ is conjugation by ρ_k for $k \neq 1$, all of the relations which do not involve $\tilde{\rho}_1$, $\tilde{\epsilon}$, and $\tilde{\Delta}$ hold since they hold in the monomial braid group. Additionally, note that the automorphisms ρ_k , ϵ , and Δ of $\overline{P}(r, n)$ generate the normalizer $N_\Gamma(\overline{P}(r, n)) = N_\Gamma(\overline{B}(r, n))$, where $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$, so they satisfy the analogs of the relations stated in the Proposition.

These observations, together with the formulas for the automorphisms $\tilde{\rho}_k$, $\tilde{\epsilon}$, $\tilde{\Delta}$, Ψ , and Υ_i recorded in §5 and (13), may be used to verify that all of the asserted relations hold. For instance, let $\tau = \rho_0 \rho_1 \cdots \rho_{n-1}$ and $\tilde{\tau} = \tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1}$. Note that $\tau^n = 1$. One can check that

$$\begin{aligned} \tilde{\tau}(C_1) &= \tau(C_1) \cdot Z_{r,n} = C_n^{W_1} \cdot Z_{r,n}, & \tilde{\tau}(C_2) &= \tau(C_2) \cdot Z_{r,n}^{-1} = C_1^{W_2} \cdot Z_{r,n}^{-1}, \\ \tilde{\tau}(C_j) &= \tau(C_j) = C_{j-1}^{W_j} \quad (3 \leq j \leq n), & \tilde{\tau}(A_{i,j}^{(q)}) &= \tau(A_{i,j}^{(q)}), \end{aligned}$$

for certain words $W_j \in P(r, n)$. This, together with the fact $\tau^n = 1$, may be used to show that $\tilde{\tau}^n = (\tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1})^n = 1$.

For the relation $\tilde{\Delta} \tilde{\rho}_0 \tilde{\Delta} = (\tilde{\rho}_{n-1} \cdots \tilde{\rho}_1 \tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1})^{-1}$, it is enough to show that $\tilde{\lambda} = \tilde{\rho}_{n-1} \cdots \tilde{\rho}_1 \tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1} \tilde{\Delta} \tilde{\rho}_0 \tilde{\Delta} = 1$. The analogous automorphism $\lambda = \rho_{n-1} \cdots \rho_1 \rho_0 \rho_1 \cdots \rho_{n-1} \Delta \rho_0 \Delta$ is trivial (consider its action on the generators of $B(r, n)$). Checking that $\tilde{\lambda}(x) = \lambda(x)$ for each generator x of $P(r, n)$ reveals that $\tilde{\lambda} = 1$ as well.

Verification of the remaining relations may be handled in a similar manner, and is left to the reader. \square

Theorem 11. *For $n \geq 3$, the automorphism group $\text{Aut}(P(r, n))$ of the pure monomial braid group admits a presentation with generators*

$$\tilde{\epsilon}, \tilde{\Delta}, \tilde{\rho}_k, \quad 0 \leq k \leq n-1, \quad \Psi, \quad \Upsilon_l, \quad 2 \leq l \leq n, \quad \Phi_{i,j,p}, \quad 1 \leq i < j \leq n, \quad 1 \leq p \leq r,$$

and relations

$$\begin{aligned} \tilde{\rho}_i \tilde{\rho}_{i+1} \tilde{\rho}_i &= \tilde{\rho}_{i+1} \tilde{\rho}_i \tilde{\rho}_{i+1} \text{ for } 1 \leq i < n, & \tilde{\rho}_i \tilde{\rho}_j &= \tilde{\rho}_j \tilde{\rho}_i \text{ for } |i - j| \geq 2, & \tilde{\epsilon}^2 &= 1, \\ (\tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1})^n &= 1, & (\tilde{\rho}_0 \tilde{\rho}_1)^2 &= (\tilde{\rho}_1 \tilde{\rho}_0)^2, & \tilde{\Delta}^2 &= 1, \\ \tilde{\Delta} \tilde{\rho}_0 \tilde{\Delta} &= (\tilde{\rho}_{n-1} \cdots \tilde{\rho}_1 \tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_{n-1})^{-1}, & (\tilde{\epsilon} \tilde{\rho}_k)^2 &= 1 \text{ for } 0 \leq k < n, & [\tilde{\epsilon}, \tilde{\Delta}] &= 1, \\ \tilde{\Delta} \tilde{\rho}_k \tilde{\Delta} &= \tilde{\rho}_{n-k} \text{ for } 1 \leq k < n, \\ \Psi^2 &= 1, \quad [\Upsilon_l, \Phi_{i,j,p}] = 1, \forall l, i, j, p, & [\Phi_{i,j,p}, \Phi_{k,l,q}] &= 1, \forall i, j, k, l, p, q, & \tilde{\epsilon} \Psi \tilde{\epsilon} &= \Psi, \\ \Psi \Upsilon_l \Psi &= \Upsilon_l^{-1}, \forall l, & \Psi \Phi_{i,j,p} \Psi &= \Phi_{i,j,p}^{-1}, \forall i, j, p, & \tilde{\Delta} \Psi \tilde{\Delta} &= \Psi, & \tilde{\rho}_k^{-1} \Psi \tilde{\rho}_k &= \Psi, \forall k, \end{aligned}$$

$$\begin{aligned}
\tilde{\Delta}\Upsilon_l\tilde{\Delta} &= \begin{cases} \Upsilon_{n-l+1}^{-1}\Upsilon_n & l < n, \\ \Upsilon_n & l = n, \end{cases} & \tilde{\epsilon}\Upsilon_l\tilde{\epsilon} &= \Upsilon_l^{-1}, \forall l, & \tilde{\rho}_0^{-1}\Upsilon_l\tilde{\rho}_0 &= \Upsilon_l, \forall l, \\
\tilde{\rho}_1^{-1}\Upsilon_l\tilde{\rho}_1 &= \begin{cases} \Upsilon_2^{-1} & l = 2, \\ \Upsilon_2^{-1}\Upsilon_l & l \neq 2, \end{cases} & \tilde{\rho}_k^{-1}\Upsilon_l\tilde{\rho}_k &= \begin{cases} \Upsilon_{k+1} & l = k, \\ \Upsilon_k & l = k+1, \\ \Upsilon_l & l \neq k, k+1, \end{cases} & \text{for } k \geq 2, \\
\tilde{\Delta}\Phi_{i,j,p}\tilde{\Delta} &= \begin{cases} \Upsilon_{n-j+1}^{-1}\Upsilon_{n-i+1}^{-1}\Upsilon_n\Phi_{n-j+1,n-i+1,p} & j < n, \\ \Upsilon_{n-i+1}^{-1}\Upsilon_n\Phi_{1,n-i+1,p} & j = n, \end{cases} \\
\tilde{\epsilon}\Phi_{i,j,p}\tilde{\epsilon} &= \begin{cases} \Phi_{i,j,r-p}^{-1} & p < r, \\ \Phi_{i,j,r}^{-1} & p = r, \end{cases} & \tilde{\rho}_0^{-1}\Phi_{i,j,q}\tilde{\rho}_0 &= \begin{cases} \Phi_{1,j,r} & i = 1, q = 1, \\ \Phi_{1,j,q-1} & i = 1, q \neq 1, \\ \Phi_{i,j,q} & \text{otherwise}, \end{cases} \\
\tilde{\rho}_1^{-1}\Phi_{i,j,q}\tilde{\rho}_1 &= \begin{cases} \Upsilon_2^{-1}\Phi_{1,2,r-p} & i = 1, j = 2, p < r, \\ \Upsilon_2^{-1}\Phi_{2,j,p} & i = 1, j > 2, \\ \Upsilon_2^{-1}\Phi_{1,j,p} & i = 2, \\ \Upsilon_2^{-1}\Phi_{i,j,p} & \text{otherwise}, \end{cases} \\
\tilde{\rho}_k^{-1}\Phi_{i,j,q}\tilde{\rho}_k &= \begin{cases} \Phi_{k,j,p} & k = i - 1, \\ \Phi_{k+1,j,p} & k = i < j - 1, \\ \Phi_{k,k+1,r-p} & k = i = j - 1, p < r, \\ \Phi_{i,k,p} & k = j - 1 > i, \\ \Phi_{i,k+1,p} & k = j, \\ \Phi_{i,j,p} & \text{otherwise} \end{cases} & \text{for } k \geq 2.
\end{aligned}$$

Proof. By Proposition 2 and Proposition 9, there is a split, short exact sequence

$$1 \rightarrow \text{tv}(P(r, n)) \rightarrow \text{Aut}(P(r, n)) \hookrightarrow N_\Gamma(\overline{P}(r, n)) \rightarrow 1,$$

where $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$. From the proof of Lemma 8, the automorphisms Ψ , Υ_l , and $\Phi_{i,j,p}$ generate the transvection subgroup $\text{tv}(P(r, n))$, and satisfy the relations $\Psi^2 = 1$, $[\Upsilon_l, \Phi_{i,j,p}] = 1$, $\Psi\Upsilon_l\Psi = \Upsilon_l^{-1}$, $\Psi\Phi_{i,j,p}^{-1}\Psi = \Phi_{i,j,p}^{-1}$. Since the automorphisms $\tilde{\epsilon}, \tilde{\Delta}, \tilde{\rho}_k$ induce the generators of $N_\Gamma(\overline{P}(r, n))$, these automorphisms, together with the aforementioned transvections, generate $\text{Aut}(P(r, n))$. By Proposition 10, the automorphisms $\tilde{\epsilon}, \tilde{\Delta}, \tilde{\rho}_k$ satisfy the relations of $N_\Gamma(\overline{P}(r, n))$. So it suffices to show that the actions of these automorphisms on the transvections Ψ , Υ_l , and $\Phi_{i,j,p}$ are as asserted. This may be accomplished by calculations with the descriptions of these automorphisms given in (15), §5, and (13). \square

Remark 12. Theorem 11 exhibits the semidirect product structure $\text{Aut}(P(r, n)) \cong \text{tv}(P(r, n)) \rtimes N_\Gamma(\overline{P}(r, n))$, where $\Gamma = \text{Mod}(\mathbb{S}_{n+2})$ and $\text{tv}(P(r, n)) \cong \mathbb{Z}^{N_r} \rtimes \mathbb{Z}_2$. Recall that $N_\Gamma(\overline{P}(r, n)) = N_\Gamma(\overline{B}(r, n)) = \overline{B}(r, n) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ (see (14)), and note that the transvection Ψ commutes with all generators of $N_\Gamma(\overline{P}(r, n))$. It follows that $\text{Aut}(P(r, n)) \cong (\mathbb{Z}^{N_r} \rtimes \overline{B}(r, n)) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

In the case $n = 2$, we have $P(r, 2) \cong \mathbb{Z} \times F_{r+1}$, and the split extension (6) yields $\text{Aut}(P(r, 2)) \cong \text{tv}(P(r, 2)) \rtimes \text{Aut}(F_{r+1})$, where $\text{tv}(P(r, 2)) \cong \mathbb{Z}^{r+1} \rtimes \mathbb{Z}_2$, generated by Ψ , Υ_2 , $\Phi_{1,2,p}$, $1 \leq p \leq r$, (see Lemma 8), and

$$F_{r+1} = \overline{P}(r, 2) = P(r, 2)/Z(P(r, 2)) = P(r, 2)/\langle Z_{r,2} \rangle = \langle A_{1,2}^{(1)}, \dots, A_{1,2}^{(r-1)}, C_2, A_{1,2}^{(r)} \rangle$$

is the free group on $r + 1 \geq 3$ generators. The group $\text{Aut}(F_{r+1})$ admits a presentation, due to Nielsen, with generators P , Q , σ , and U , where

$$\begin{aligned} P: & \begin{cases} A_{1,2}^{(1)} \mapsto A_{1,2}^{(2)}, \\ A_{1,2}^{(2)} \mapsto A_{1,2}^{(1)}, \\ A_{1,2}^{(q)} \mapsto A_{1,2}^{(q)} & q \neq 1, 2, \\ C_2 \mapsto C_2, \end{cases} & Q: & \begin{cases} A_{1,2}^{(q)} \mapsto A_{1,2}^{(q+1)} & q < r-1, \\ A_{1,2}^{(r-1)} \mapsto C_2, \\ A_{1,2}^{(r)} \mapsto A_{1,2}^{(1)}, \\ C_2 \mapsto A_{1,2}^{(r)}, \end{cases} \\ \sigma: & \begin{cases} A_{1,2}^{(1)} \mapsto (A_{1,2}^{(1)})^{-1}, \\ A_{1,2}^{(q)} \mapsto A_{1,2}^{(q)} & q \neq 1, \\ C_2 \mapsto C_2, \end{cases} & U: & \begin{cases} A_{1,2}^{(1)} \mapsto A_{1,2}^{(1)} A_{1,2}^{(2)}, \\ A_{1,2}^{(q)} \mapsto A_{1,2}^{(q)} & q \neq 1, \\ C_2 \mapsto C_2, \end{cases} \end{aligned}$$

Refer to [MKS66, §3.5, Cor. N1] for the relations satisfied by these generators of $\text{Aut}(F_{r+1})$. Lifts of these automorphisms to automorphisms of $P(r, 2)$ which fix the generator $Z_{r,2} = C_1 A_{1,2}^{(1)} \cdots A_{1,2}^{(r-1)} C_2 A_{1,2}^{(r)}$ of the center are given by setting

$$\begin{aligned} P(C_1) &= C_1 [A_{1,2}^{(1)}, A_{1,2}^{(2)}], & Q(C_1) &= (A_{1,2}^{(1)})^{-1} C_1 A_{1,2}^{(1)}, \\ \sigma(C_1) &= C_1 (A_{1,2}^{(1)})^2, & U(C_1) &= C_1 A_{1,2}^{(1)} (A_{1,2}^{(1)} A_{1,2}^{(2)})^{-1}. \end{aligned}$$

Calculations with these formulas yields the following result.

Proposition 13. *The automorphism group of the pure monomial braid group $P(r, 2)$ is generated by P , Q , σ , U , Ψ , Υ_2 , $\Phi_{1,2,p}$, $1 \leq p \leq r$, and decomposes as a semidirect product $\text{Aut}(P(r, 2)) \cong \text{tv}(P(r, n)) \rtimes \text{Aut}(F_{r+1})$. The action of $\text{Aut}(F_{r+1})$ on $\text{tv}(P(r, n))$ is given by*

$$\begin{aligned} P: & \begin{cases} \Psi \mapsto \Psi, \\ \Upsilon_2 \mapsto \Upsilon_2, \\ \Phi_{1,2,1} \mapsto \Phi_{1,2,2}, \\ \Phi_{1,2,2} \mapsto \Phi_{1,2,1}, \\ \Phi_{1,2,p} \mapsto \Phi_{1,2,p} & p \neq 1, 2, \end{cases} & Q: & \begin{cases} \Psi \mapsto \Psi, \\ \Upsilon_2 \mapsto \Phi_{1,2,r}, \\ \Phi_{1,2,p} \mapsto \Phi_{1,2,p+1} & p \leq r-2, \\ \Phi_{1,2,r-1} \mapsto \Upsilon_2, \\ \Phi_{1,2,r} \mapsto \Phi_{1,2,1}, \end{cases} \\ \sigma: & \begin{cases} \Psi \mapsto \Psi, \\ \Upsilon_2 \mapsto \Upsilon_2, \\ \Phi_{1,2,1} \mapsto \Phi_{1,2,1}^{-1}, \\ \Phi_{1,2,p} \mapsto \Phi_{1,2,p} & p \neq 1, \end{cases} & U: & \begin{cases} \Psi \mapsto \Psi, \\ \Upsilon_2 \mapsto \Upsilon_2, \\ \Phi_{1,2,2} \mapsto \Phi_{1,2,1}^{-1} \Phi_{1,2,2}, \\ \Phi_{1,2,p} \mapsto \Phi_{1,2,p} & p \neq 2. \end{cases} \end{aligned}$$

Remark 14. Note that the generator Ψ of $\text{tv}(P(r, 2)) < \text{Aut}(P(r, 2))$ commutes with the generators P, Q, σ, U of $\text{Aut}(P(r, 2))$ which project to the generators of $\text{Aut}(F_{r+1})$. It follows that $\text{Aut}(P(r, 2)) \cong \mathbb{Z}^{r+1} \rtimes (\mathbb{Z}_2 \times \text{Aut}(F_{r+1}))$.

Since $P(r, 1) = \mathbb{Z}$ is infinite cyclic, $\text{Aut}(P(r, 1)) = \mathbb{Z}_2$.

5. THE ACTION OF $\text{Aut}(P(r, n))$ ON THE GENERATORS OF $P(r, n)$

We record the action of the elements $\tilde{\epsilon}, \tilde{\Delta}, \tilde{\rho}_k, 0 \leq k \leq n-1$, of $\text{Aut}(P(r, n))$ on the generators $C_j, 1 \leq j \leq n$, and $A_{i,j}^{(q)}, 1 \leq i < j \leq n, 1 \leq q \leq r$ of the pure monomial braid group $P(r, n)$. See (13) for the action of the generators of the transvection subgroup $\text{tv}(P(r, n))$. Recall the elements $A_{i,j}^{[q]}, V_{i,j}^{(q)}, D_k \in P(r, n)$ from (8), and

that $y^x = x^{-1}yx$. For $1 \leq i < j \leq n$ and $1 \leq q \leq r$, let $U_i^{(q)} = A_{i,i+1}^{(q)} A_{i,i+2}^{(q)} \cdots A_{i,n}^{(q)}$. The actions of $\tilde{\epsilon}$, $\tilde{\Delta}$, and $\tilde{\rho}_k$ are given by:

$$\begin{aligned} \tilde{\epsilon}: & \begin{cases} C_j \mapsto C_1^{-1} Z_{r,n}^2 & \text{if } j = 1, \\ C_j \mapsto (V_{1,j}^{(r)})^{-1} C_j^{-1} V_{1,j}^{(r)} & \text{if } j \neq 1, \\ A_{i,j}^{(q)} \mapsto (V_{i+1,j}^{(r)})^{-1} (A_{i,j}^{(r)})^{-1} V_{i+1,j}^{(r)} & \text{if } q = r, \\ A_{i,j}^{(q)} \mapsto (V_{i+1,j}^{(r)})^{-1} D_i (A_{i,j}^{(r-q)})^{-1} D_i^{-1} V_{i+1,j}^{(r)} & \text{if } q \neq r, \end{cases} \\ \tilde{\Delta}: & \begin{cases} C_j \mapsto D_n^{-1} Z_{r,n} & \text{if } j = 1, \\ C_j \mapsto [U_{n-j+1}^{(r)} D_{n-j+1} U_{n-j+1}^{(1)} \cdots U_{n-j+1}^{(r-1)}]^{-1} & \text{if } j \neq 1, n, \\ C_j \mapsto [U_1^{(r)} D_1 U_1^{(1)} \cdots U_1^{(r-1)}]^{-1} Z_{r,n} & \text{if } j = n, \\ A_{i,j}^{(q)} \mapsto (A_{n-j+1,n-i+1}^{(r)})^{V_{n-j+1,n-i+1}^{(r)}} & \text{if } q = r, \\ A_{i,j}^{(q)} \mapsto (A_{n-j+1,n-i+1}^{(r)})^{V_{n-j+1,n-i+1}^{(r)} D_{n-j+1}^{-1} D_{n-i+1}^{-1}} & \text{if } q \neq r, \end{cases} \\ \tilde{\rho}_0: & \begin{cases} C_j \mapsto C_1 & \text{if } j = 1, \\ C_j \mapsto C_j^{(A_{1,j}^{(r-1)})^{-1}} & \text{if } j \neq 1, \\ A_{i,j}^{(q)} \mapsto A_{i,j}^{(q-1)} & \text{if } i = 1, q \neq 1, \\ A_{i,j}^{(q)} \mapsto (A_{i,j}^{(r)})^{C_1} & \text{if } i = 1, q = 1, \\ A_{i,j}^{(q)} \mapsto A_{i,j}^{(q)} & \text{if } i \geq 2, \end{cases} \\ \tilde{\rho}_1: & \begin{cases} C_j \mapsto C_2^{A_{1,2}^{(r)}} Z_{r,n} & \text{if } j = 1, \\ C_j \mapsto C_1 Z_{r,n}^{-1} & \text{if } j = 2, \\ C_j \mapsto C_j & \text{if } j \geq 3, \\ A_{i,j}^{(q)} \mapsto (A_{1,2}^{(r-q)})^{(C_1 A_{1,2}^{(1)} \cdots A_{1,2}^{(r-q-1)})^{-1}} & \text{if } i = 1, j = 2, q < r, \\ A_{i,j}^{(q)} \mapsto A_{2,j}^{(q)} & \text{if } i = 1, j \geq 3, q = r, \\ A_{i,j}^{(q)} \mapsto (A_{2,j}^{(q)})^{(C_1 A_{1,2}^{(1)} \cdots A_{1,2}^{(r-q-1)} C_1^{-1})^{-1}} & \text{if } i = 1, j \geq 3, q < r, \\ A_{i,j}^{(q)} \mapsto (A_{1,2}^{(q)})^{A_{1,2}^{(q+1)} \cdots A_{1,2}^{(r)}} & \text{if } i = 2, q < r, \\ A_{i,j}^{(q)} \mapsto A_{i,j}^{(q)} & \text{otherwise,} \end{cases} \\ \tilde{\rho}_k: & \begin{cases} C_j \mapsto C_{j-1} & \text{if } k = j - 1, \\ C_j \mapsto C_{j+1}^{A_{j,j+1}^{(r)}} & \text{if } k = j, \\ C_j \mapsto C_j & \text{if } k \neq j - 1, j, \\ A_{i,j}^{(q)} \mapsto (A_{i-1,j}^{(q)})^{A_{i-1,i}^{[q]} A_{i-1,i}^{(r)}} & \text{if } k = i - 1, q < r, \\ A_{i,j}^{(q)} \mapsto A_{i-1,j}^{(r)} & \text{if } k = i - 1, q = r, \\ A_{i,j}^{(q)} \mapsto (A_{i+1,j}^{(q)})^{(D_i A_{i,i+1}^{(1)} \cdots A_{i,i+1}^{(r-q-1)} D_i^{-1})^{-1}} & \text{if } k = i < j - 1, q < r, \\ A_{i,j}^{(q)} \mapsto A_{i+1,j}^{(r)} & \text{if } k = i < j - 1, q = r, \\ A_{i,j}^{(q)} \mapsto (A_{i,i+1}^{(r-q)})^{(D_i A_{i,i+1}^{(1)} \cdots A_{i,i+1}^{(r-q-1)})^{-1}} & \text{if } k = i = j - 1, q < r, \\ A_{i,j}^{(q)} \mapsto A_{i,j-1}^{(q)} & \text{if } k = j - 1 > i, \\ A_{i,j}^{(q)} \mapsto (A_{i,j+1}^{(q)})^{A_{j,j+1}^{(r)}} & \text{if } k = j, \\ A_{i,j}^{(q)} \mapsto A_{i,j}^{(q)} & \text{otherwise,} \end{cases} \end{aligned}$$

for $2 \leq k \leq n - 1$.

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